Linear systems – Midterm exam – Solutions

Midterm exam 2024–2025, Thursday 5 June 2025, 9:00 – 11:00

Instructions

1. The use of books and lecture notes is not allowed, but you can use a one-page cheat sheet.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1

(6 + 10 + 8 + 10 = 34 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \text{with} \qquad A = \begin{bmatrix} 5 & 1 & -1 \\ -14 & -3 & 3 \\ 16 & 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}. \tag{1}$$

(a) Show that the system is not controllable.

(b) Find a nonsingular matrix T and matrices A_{11}, A_{12}, A_{22}, and B_1 such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where the matrix pair (A_{11}, B_1) is controllable.

Hint. It is sufficient to give T^{-1} instead of T, but please give A_{11}, A_{12}, A_{22}, and B_1.

(c) Use the result of (b) to show that the system (1) is stabilizable.

(d) Use the matrix T from (b) to find a matrix F such that the feedback $u(t) = Fx(t)$ stabilizes the system (1).

Answer Problem 1(a)

To verify controllability, compute

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}. \tag{2}$$

Hence,

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix} = 1, \tag{3}$$

which is smaller than the state-space dimension (which is 3), i.e., the system is not controllable.

Answer Problem 1(b)

Note that

$$\operatorname{im}\begin{bmatrix}B & AB & A^2B\end{bmatrix} = \operatorname{im}\begin{bmatrix}1 & 1 & 1\\ -2 & -2 & -2\\ 2 & 2 & 2\end{bmatrix} = \operatorname{span}\{q_1\} \tag{4}$$

for

$$q_1 = \begin{bmatrix}1\\ -2\\ 2\end{bmatrix}. \tag{5}$$

In other words, $\operatorname{span}\{q_1\}$ is a basis for the reachable subspace $\mathcal{W}$. To find the desired coordinate transformation, we need to extend this basis to a basis of $\mathbb{R}^3$, i.e., we need to choose $q_2, q_3 \in \mathbb{R}^3$ such that

$$\operatorname{span}\{q_1, q_2, q_3\} = \mathbb{R}^3. \tag{6}$$

Then, we choose T such that

$$T^{-1} = \begin{bmatrix}q_1 & q_2 & q_3\end{bmatrix}. \tag{7}$$

Here, we choose

$$q_2 = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \qquad q_3 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, \tag{8}$$

which leads to

$$T^{-1} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 0 & 1\end{bmatrix} \quad \Longleftrightarrow \quad T = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -2 & 0 & 1\end{bmatrix} \tag{9}$$

and

$$TAT^{-1} = \begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 1\\ 0 & 1 & -2\end{bmatrix}, \qquad TB = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}. \tag{10}$$

Thus, we have in particular that

$$A_{11} = 1, \quad A_{12} = \begin{bmatrix}1 & -1\end{bmatrix}, \quad A_{22} = \begin{bmatrix}-1 & 1\\ 1 & -2\end{bmatrix}, \quad B_1 = 1. \tag{11}$$

An alternative natural choice would be to take

$$q_2 = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}, \qquad q_3 = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \tag{12}$$

yielding

$$T^{-1} = \begin{bmatrix}1 & 1 & 0\\ -2 & 0 & 1\\ 2 & 0 & 0\end{bmatrix} \quad \Longleftrightarrow \quad T = \frac{1}{2}\begin{bmatrix}0 & 0 & 1\\ 2 & 0 & -1\\ 0 & 2 & 2\end{bmatrix} \tag{13}$$

and

$$TAT^{-1} = \begin{bmatrix}1 & 8 & 1.5\\ 0 & -3 & -0.5\\ 0 & 2 & 0\end{bmatrix}, \qquad TB = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}. \tag{14}$$

Answer Problem 1(c)

A sufficient condition for stabilizability is that $\sigma(A_{22}) \subset \mathbb{C}_-$. To evaluate this, we consider A_{22} from (11) and compute its characteristic polynomial as

$$\Delta_{A_{22}}(s) = \det(sI - A_{22}) = \begin{vmatrix} s+1 & -1 \\ -1 & s+2 \end{vmatrix} = (s+1)(s+2) - 1 = s^2 + 3s + 1. \tag{15}$$

As all coefficients of this polynomial of degree two are nonzero and positive (have the same sign), the polynomial is stable and $\sigma(A_{22}) \subset \mathbb{C}_-$. Thus, the system is stabilizable.

Answer Problem 1(d)

The use of the feedback $u(t) = Fx(t)$ to the system (1) leads to the closed-loop system

$$\dot{x}(t) = (A + BF)x(t), \tag{16}$$

and the stabilization problem asks to find F such that

$$\sigma(A + BF) \subset \mathbb{C}_-. \tag{17}$$

By similarity transformation, we know that

$$\sigma(A + BF) = \sigma\big(T(A + BF)T^{-1}\big) \tag{18}$$

for any nonsingular T. We choose T as in (9) to obtain

$$T(A + BF)T^{-1} = TAT^{-1} + TBFT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} FT^{-1} = \begin{bmatrix} A_{11} + B_1F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{19}$$

where the final equality is obtained by choosing

$$FT^{-1} = \begin{bmatrix} F_1 & 0 \end{bmatrix}. \tag{20}$$

From (18) and the block upper triangular structure in (19), we have that

$$\sigma(A + BF) = \sigma(A_{11} + B_1F_1) \cup \sigma(A_{22}). \tag{21}$$

Recalling that $\sigma(A_{22}) \subset \mathbb{C}_-$ from (c), it is sufficient to choose F_1 to satisfy

$$\sigma(A_{11} + B_1F_1) \subset \mathbb{C}_-. \tag{22}$$

Using $A_{11} = 1$ and $B_1 = 1$ from (11), this is true for any $F_1 < -1$. Finally, in the original coordinates, we have

$$F = \begin{bmatrix} F_1 & 0 \end{bmatrix} T = \begin{bmatrix} F_1 & 0 & 0 \end{bmatrix}, \tag{23}$$

as follows from T in (9).

Problem 2 (18 points)

Find all values of $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -2 & -a & -1 \end{bmatrix} x(t)$$

is asymptotically stable.

Answer Problem 2

First, note that the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -2 & -a & -1 \end{bmatrix} \tag{24}$$

is in companion form. As a consequence, its characteristic polynomial is easily read as

$$\Delta_A(s) = s^4 + s^3 + as^2 + 2s + b. \tag{25}$$

We now proceed stability analysis through the following Routh-Hurwitz table:

		s^4	s^3	s^2	s^1	s^0	
$1\times$	Δ_A	1	1	a	2	b	(step 0)
$1\times$		1	0	2	0		
$(a-2)\times$	q		1	$a-2$	2	b	(step 1)
$1\times$			$a-2$	0	b	0	
	r			$(a-2)^2$	$2(a-2)-b$	$(a-2)b$	(step 2)

Step 0. Considering the polynomial Δ_A, and recalling that a necessary condition for stability is that all coefficients have the same sign, we require

$$a > 0, \qquad b > 0. \tag{26}$$

Then, noting that the two leading coefficients of Δ_A (both equaling 1) are nonzero and have the same sign, we proceed by constructing the polynomial q as given in the table.

Step 1. Following the same reasoning as before, a necessary condition for stability of q is that

$$a - 2 > 0, \tag{27}$$

where we recall that we already had $b > 0$ from Step 0. Under the assumption $a - 2 > 0$, we again have that the two leading coefficients of the polynomial q are nonzero and have the same sign, allowing for taking an additional step in the table, leading to the polynomial r.

Step 2. The polynomial r is of degree 2, meaning that it is stable if and only if all its coefficients are nonzero and have the same sign. Under the assumptions (26) and (27), this is equivalent to stating that

$$2(a-2) - b > 0. \tag{28}$$

Conclusion. Collecting results, we have that (26), (27), and (28) are necessary for stability of Δ_A, by the Routh-Hurwitz procedure. These conditions can be summarized as

$$2 < a, \qquad 0 < b < 2(a-2), \tag{29}$$

and we note that these conditions are also sufficient for stability of r. By the Routh-Hurwitz procedure, this in turn means that these conditions are sufficient for stability for q and Δ_A, subsequently. Hence, (29) is necessary and sufficient for stability of Δ_A.

Finally, this means that (29) is necessary and sufficient for asymptotic stability of the system.

Problem 3 (12 + 8 + 8 + 10 = 38 points)

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Gd(t), \\ y(t) &= Cx(t), \end{aligned} \tag{30}$$

with state $x(t) \in \mathbb{R}^n$, external disturbance $d(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^p$. For initial condition $x(0) = x_0$ and disturbance $d : [0, \infty) \to \mathbb{R}^m$, we denote by $y(t; x_0, d)$ the corresponding output solution, i.e.,

$$y(t; x_0, d) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Gd(\tau)\,\mathrm{d}\tau.$$

We say that the system (30) is *disturbance decoupled* if the disturbance d does not affect the output solution, i.e., $y(t, x_0, d_1) = y(t, x_0, d_2)$ for all $t \geq 0$, for all $x_0 \in \mathbb{R}^n$ and all d_1, d_2.

(a) Show that the system is disturbance decoupled if and only if

$$Ce^{At}G = 0 \text{ for all } t \geq 0. \tag{31}$$

Next, consider the following statements:

(i) $CA^kG = 0$ for $k = 0, 1, 2, \ldots$.

(ii) there exists an A-invariant subspace $\mathcal{V} \subset \mathbb{R}^n$ such that $\operatorname{im} G \subset \mathcal{V} \subset \ker C$.

Note that (31) is equivalent to (i), meaning that the system is disturbance decoupled if and only if (i) holds. In the remainder of this problem, we will show (i) $\iff$ (ii).

(b) Let $\mathcal{V} \subset \mathbb{R}^n$ be an A-invariant subspace satisfying $\operatorname{im} G \subset \mathcal{V}$. Show that

$$\operatorname{im} A^kG \subset \mathcal{V} \text{ for } k = 0, 1, 2, \ldots.$$

(c) Use the result from (b) to show that (ii) implies (i).

(d) Show that (i) implies (ii).

Answer Problem 3(a)

By definition, the system (30) is disturbance decoupled if

$$Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Gd_1(\tau)\,\mathrm{d}\tau = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Gd_2(\tau)\,\mathrm{d}\tau \quad \text{for all } t \geq 0, \tag{32}$$

for all initial conditions $x_0 \in \mathbb{R}^n$ and all disturbances d_1, d_2. By defining $d = d_1 - d_2$, this is equivalent to the condition

$$0 = \int_0^t Ce^{A(t-\tau)}Gd(\tau)\,\mathrm{d}\tau \quad \text{for all } t \geq 0, \tag{33}$$

for all disturbance functions d. Note that the initial condition $x_0 \in \mathbb{R}^n$ does not play a role.

Proof of if. If (31) holds, it is clear that (33) holds for any $d : [0, \infty) \to \mathbb{R}^m$, meaning that the system is disturbance decoupled.

Proof of only if. Let the system be disturbance decoupled, i.e., (33) holds, and fix some time $t \geq 0$. As (33) holds for any disturbance d, it also holds for the particular disturbance $d : [0, t] \to \mathbb{R}^m$ defined as

$$d(s) = G^{\mathrm{T}} e^{A^{\mathrm{T}}(t-s)} C^{\mathrm{T}} v, \tag{34}$$

where $v \in \mathbb{R}^p$ is an arbitrary vector. Substitution in (33) gives

$$0 = \int_0^t C e^{A(t-\tau)} G G^{\mathrm{T}} e^{A^{\mathrm{T}}(t-\tau)} C^{\mathrm{T}} v \, \mathrm{d}\tau, \tag{35}$$

after which pre-multiplication with v^{T} yields

$$0 = \int_0^t v^{\mathrm{T}} C e^{A(t-\tau)} G G^{\mathrm{T}} e^{A^{\mathrm{T}}(t-\tau)} C^{\mathrm{T}} v \, \mathrm{d}\tau = \int_0^t \left| G^{\mathrm{T}} e^{A^{\mathrm{T}}(t-\tau)} C^{\mathrm{T}} v \right|^2 \mathrm{d}\tau. \tag{36}$$

Here, $|\cdot|$ denotes the Euclidean norm. A consequence of this result is that the function $\tau \mapsto G^{\mathrm{T}} e^{A^{\mathrm{T}}(t-\tau)} C^{\mathrm{T}} v$ equals zero on the interval $[0, t]$. This in fact holds for any $t \geq 0$, such that

$$G^{\mathrm{T}} e^{A^{\mathrm{T}} t} C^{\mathrm{T}} v = 0 \quad \text{for all } t \geq 0. \tag{37}$$

As this result holds for any choice of $v \in \mathbb{R}^p$, this implies (after transposing the result) that

$$C e^{At} G = 0 \quad \text{for all } t \geq 0. \tag{38}$$

Answer Problem 3(b)

As a first step, we show that A-invariance of $\mathcal{V}$ implies that, for all $k = 0, 1, 2, \ldots$,

$$v \in \mathcal{V} \quad \Longrightarrow \quad A^k v \in \mathcal{V}. \tag{39}$$

We prove this by induction. First, it is clear that the above holds for $k = 0$. To prove the inductive step, assume that (39) holds for some $k \in \{0, 1, 2, \ldots\}$. However, as $A^k v \in \mathcal{V}$, we have by A-invariance of $\mathcal{V}$ that $AA^k v = A^{k+1} v \in \mathcal{V}$, which gives the desired result.

Now, we take $k \in \{0, 1, 2, \ldots\}$. To show that $\operatorname{im} A^k G \subset \mathcal{V}$, let $w \in \operatorname{im} A^k G$. In other words, there exists d such that

$$w = A^k G d. \tag{40}$$

Denote $v = Gd$. Noting that $\operatorname{im} G \subset \mathcal{V}$ by assumption, we have $v \in \mathcal{V}$, after which (39) shows that $A^k v = A^k G d = w \in \mathcal{V}$. As the choice w was arbitrary, this proves the desired result.

Answer Problem 3(c)

We assume that (ii) holds, i.e., there exists an A-invariant subspace $\mathcal{V} \subset \mathbb{R}^n$ such that $\operatorname{im} G \subset \mathcal{V} \subset \ker C$. By (b), we know that, for any $k \in \{0, 1, \ldots\}$,

$$\operatorname{im} A^k G \subset \mathcal{V}. \tag{41}$$

We fix some $k \in \{0, 1, \ldots\}$ and note that $A^k G d \in \operatorname{im} A^k G$ for any $d \in \mathbb{R}^m$. Then, as $\mathcal{V} \subset \ker C$, this implies

$$A^k G d \in \ker C \quad \Longleftrightarrow \quad C A^k G d = 0. \tag{42}$$

In other words, $CA^k G d = 0$ for any $d \in \mathbb{R}^m$, which implies $CA^k G = 0$. This proves the result.

Answer Problem 3(d)

Define

$$\mathcal{W} = \operatorname{im} \begin{bmatrix} G & AG & \cdots & A^{n-1}G \end{bmatrix}. \tag{43}$$

Noting that $\mathcal{W}$ is the reachable subspace for (30) (when d is regarded as an input rather than a disturbance), we have that $\mathcal{W}$ is A-invariant and contains $\operatorname{im} G$. Hence, it remains to show that $\mathcal{W} \subset \ker C$. To do so, let $w \in \mathcal{W}$, i.e., there exists vectors $d_0, d_1, \ldots, d_{n-1} \in \mathbb{R}^m$ such that

$$w = Gd_0 + AGd_1 + \cdots + A^{n-1}Gd_{n-1}. \tag{44}$$

Then,

$$Cw = CGd_0 + CAGd_1 + \cdots + CA^{n-1}Gd_{n-1} = 0, \tag{45}$$

where we have used (i). In other words, $w \in \ker C$, which concludes the proof.

(10 points free)